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THE ALGEBRAIC SOLUTION OF EQUATIONS.

By PROF. A. M. SAWIN, Laramie, Wyo.

Let the general equation $f(x) = 0$ represent

$$x^n + A\,x^{n-2} + B\,x^{n-3} + C\,x^{n-4} + \ldots + Tx + U = 0.$$

We shall denote the sum of all the terms of a given kind by Σ; thus,

$$u^2 + v^2 + y^2 + \ldots = \Sigma\,u^2, \qquad uv + uy + uz + \ldots = \Sigma\,uv.$$

We may then write $f(x) = 0$

$$x^n + [a\,\Sigma u^2 + b\,\Sigma\,uv]\,x^{n-2} + [c\,\Sigma u^3 + d\,\Sigma u^2 v + e\,\Sigma uvy]\,x^{n-3} + \ldots$$
$$+ l\,\Sigma u^n + m\,\Sigma u^{n-1} v + n\,\Sigma u^{n-2} v^2 + \ldots = 0\,,$$

in which the functions of $u, v, y, \ldots, z$ are symmetrical, and each coefficient involves all the possible symmetrical functions of $u, v, y, \ldots, z$ of its order. Further, the order of the coefficients is always such that the terms of the equation shall be homogeneous of the nth degree.

Letting $x = u + v + y + \ldots + z$, we may then determine the values of the constants $a, b, c, \ldots, l, m$ so that $u + v + y + \ldots + z$ shall be a root, and the equation shall reduce identically to zero. Let us examine the case of the cubic equation

$$x^3 + Ax + B = 0\,, \tag{1}$$

in which we have

$$x^3 + [a\,\Sigma u^2 + b\,uv]\,x + c\,\Sigma u^3 + d\,\Sigma u^2 v = 0, \tag{2}$$

there being only two variables u and v employed, because there are only two coefficients A and B that are independent. In order that $u + v$ shall be a root, we shall have identically by substituting in (2),

$$a + c + 1 = 0$$
$$a + b + d + 3 = 0.$$

Of the four quantities a, b, c, d in these adjunctive equations, as we shall term them, any two may be made zero, or assigned any values at pleasure; provided only, that in assigning zero values no coefficient be destroyed which

would have the effect to impair the generality of the equation. For instance, if a and b were simultaneously made zero, the term containing x would vanish.

We may assume $a = 0,\ d = 0$; whence $c = -1,\ b = -3$:

or $b = 0,\ d = 0$; " $a = -3,\ c = 2$:

or $b = 0,\ c = 0$; " $a = -1,\ d = -2$.

Writing the general equation (2) with these values of the constants substituted in their order, we have

$$x^3 - 3uv\,x - (u^3 + v^3) = 0\,, \tag{3}$$

$$x^3 - 3(u^2 + v^2)\,x + 2(u^3 + v^3) = 0\,, \tag{4}$$

$$x^3 - (u^2 + v^2)\,x - 2(u^2v + uv^2) = 0\,. \tag{5}$$

Any one of these equations is identically satisfied by substituting $u + v$ for x since $u + v$ is a root. Equating the first of these with (1), evidently,

$$u^3 + v^3 = -B, \qquad uv = -\tfrac{1}{3}A\,.$$

Cubing the second of these and eliminating u, we have the sextic

$$v^6 + Bv^3 - \tfrac{1}{27}A^3 = 0\,,$$

from which

$$x = u + v = \sqrt[3]{-\tfrac{1}{2}B + \sqrt{\tfrac{1}{4}B^2 + \tfrac{1}{27}A^3}} + \sqrt[3]{-\tfrac{1}{2}B - \sqrt{\tfrac{1}{4}B^2 + \tfrac{1}{27}A^3}}$$
$$= \theta_1 + \theta_2.$$

These radical functions θ_1, θ_2 will be readily recognized as Cardan's formulæ, although obtained in a different manner; and, for this reason, we may refer to form (3) as Cardan's form.

Equating (4) with (1), we have

$$u^2 + v^2 = -\tfrac{1}{3}A\,, \qquad u^3 + v^3 = -\tfrac{1}{2}B. \tag{6}$$

Eliminating u, we have the sextic

$$v^6 + \tfrac{1}{2}Av^4 - \tfrac{1}{2}Bv^3 + \tfrac{1}{6}A^2v^2 + \tfrac{1}{2}(\tfrac{1}{4}B^2 + \tfrac{1}{27}A^3) = 0\,, \tag{7}$$

which is neither a cubic nor quadratic in form, but rather a combination of these. Now u and v in these equations do not represent the same radical functions that they do in Cardan's form; namely θ_1 and θ_2, but their sum is still equal to x. So, if u is greater than θ_1, then v is less than θ_2 by the same amount, and we may at once write

$$\left.\begin{aligned} u &= \theta_1 + a\,, \\ v &= \theta_2 - a\,; \end{aligned}\right\} \tag{8}$$

and substituting these values in (6), we obtain directly

$$a^2 + (\theta_1 - \theta_2)\, a + \tfrac{1}{2}(\theta_1^2 + \theta_2^2) = \tfrac{1}{2}\,\theta_1\,\theta_2\,,$$

$$3(\theta_1 + \theta_2)\, a^2 + 3(\theta_1^2 - \theta_2^2)\, a + (\theta_1^3 + \theta_2^3) = -\tfrac{1}{2}(\theta_1^3 + \theta_2^3)\,.$$

Solving either of these quadratics in a, we obtain at once

$$a = -\tfrac{1}{2}(\theta_1 - \theta_2) \pm \sqrt{-\tfrac{1}{4}(\theta_1^2 + \theta_2^2)}\,.$$

Substituting this value in (8), using plus sign,

$$u = \tfrac{1}{2}(\theta_1 + \theta_2) + \sqrt{-\tfrac{1}{4}(\theta_1^2 + \theta_2^2)}\,,$$

$$v = \tfrac{1}{2}(\theta_1 + \theta_2) - \sqrt{-\tfrac{1}{4}(\theta_1^2 + \theta_2^2)}\,.$$

These values of u and v are algebraic solutions of the curious group (6) which produced upon elimination the irreducible sextic (7).

The six roots of the sextic in Cardan's form are obviously θ_1, $\omega\theta$, $\omega^2\theta_1$, θ_2, $\omega\theta_2$, $\omega^2\theta_2$; and taken in pairs, furnish the three roots of the cubic, as follows:

$$\begin{aligned} x = u + v &= \theta_1 + \theta_2\,, \\ \text{`` \quad ``} \quad &= \omega\theta_1 + \omega^2\theta_2\,, \\ \text{`` \quad ``} \quad &= \omega^2\theta_1 + \omega\theta_2\,; \end{aligned}$$

in which ω and ω^2 denote as usual the imaginary roots of $\omega^3 - 1 = 0$. The remaining four roots of the sextic (7) are obviously derived from the above values of u and v by simultaneously substituting $\omega\theta_1$, $\omega^2\theta_2$ and $\omega^2\theta_1$, $\omega\theta_2$ in those radical functions.

If we now equate (5) with (1), we shall have

$$u^2 + v^2 = -A\,, \qquad u^2v + uv^2 = -\tfrac{1}{2}B\,;$$

which will give a different sextic still, whose six roots may be as readily written as in the case of the sextic preceding.

Proceeding to the equation of the fourth degree

$$x^4 + A\,x^2 + B\,x + C = 0\,, \tag{9}$$

we shall have the corresponding general equation in x, u, v, and y

$$x^4 + [a\,\Sigma u^2 + b\,\Sigma uv]\,x^2 + [c\,\Sigma u^3 + d\,\Sigma u^2v + e\,uvy]\,x + f\,\Sigma u^4$$
$$+\, g\,\Sigma u^3v + h\,\Sigma u^2v^2 + j\,\Sigma u^2vy = 0\,.$$

Substituting for x the value $u+v+y$, we have

$$\begin{aligned} a+c+f+1&=0\,, \\ 2a+b+c+d+g+4&=0\,, \\ 2a+2b+2d+h+6&=0\,, \\ 2a+5b+2d+e+j+12&=0\,. \end{aligned}$$

Five of the nine constants obtained in this set of adjunctive equations may be made zero, or assigned values at pleasure. The number of groups of nine things, taking five at a time, is 126; but a few of these would have to be rejected, in order to preserve the quartic in its general form. We may write a few of these forms, assigning zero always to five of the constants.

Assume a, c, d, g, and j equal zero: evidently

$$f=-1,\quad b=-4,\quad h=2,\quad e=8\,;$$

and we have

$$\begin{aligned} x^4-4\,(uv+uy+vy)\,x^2+8\,uvy\,x-(u^4+v^4+y^4)& \\ +2\,(u^2v^2+u^2y^2+v^2y^2)&=0\,. \qquad (10) \end{aligned}$$

Assume b, c, d, g, j equal zero: evidently

$$a=-2,\quad e=-8,\quad f=1,\quad h=-2\,;$$

and we have

$$\begin{aligned} x^4-2\,(u^2+v^2+y^2)\,x^2-8\,uvy\,x+(u^4+v^4+y^4)& \\ -2\,(u^2v^2+u^2y^2+v^2y^2)&=0\,. \qquad (11) \end{aligned}$$

If a, c, d, h, j equal zero,

$$f=-1\,,\quad b=-3\,,\quad g=-1\,,\quad e=3\,;$$

and we have

$$\begin{aligned} x^4-3\,(uv+uy+vy)\,x^2+3\,uvy\,x-(u^4+v^4+y^4)& \\ -3\,(u^3v+u^3y+uv^3+uy^3+v^3y+vy^3)&=0\,. \end{aligned}$$

We may readily write the remaining forms of the quartic, subject to the restriction mentioned above in the case of the cubic, always assigning zero values to five of the constants. The number of forms would be the same, however, if the constants have no particular values assigned, but be left simply indeterminate. All these forms are theoretically important, as we shall see.

Equating the form (11) with (9), evidently,

$$-2(u^2+v^2+y^2)=A, \quad -8\,uvy=B, \quad u^4+v^4+y^4-2(u^2v^2+u^2y^2+v^2y^2)=C;$$

whence

$$u^2+v^2+y^2=-\tfrac{1}{2}A,$$
$$u^2v^2+u^2y^2+v^2y^2=\tfrac{1}{16}A^2-\tfrac{1}{4}C,$$
$$u^2v^2y^2=\tfrac{1}{64}B^2.$$

From which we have

$$x=u+v+y=\theta_1+\theta_2+\theta_3,$$

where θ_1, θ_2, θ_3 are radical functions of the coefficients expressed by solving the cubic equation,

$$\lambda^6+\tfrac{1}{2}A\lambda^4+\tfrac{1}{16}(A^2-4C)\lambda^2-\tfrac{1}{64}B^2=0.$$

This cubic is the same as employed by Euler in his reduction of the equation of the fourth degree, and we may refer to (11) as Euler's form.

If, however, we equate (10) and (9) we obtain

$$uv+uy+vy=-\tfrac{1}{4}A,$$
$$u^4+v^4+y^4-2(u^2v^2+u^2y^2+v^2y^2)=-C,$$
$$uvy=\tfrac{1}{8}B.$$

Eliminating u and v from these equations we shall have in general an equation of the 24th degree. It is obvious that the values of u, v, and y are not the same here as in Euler's form, namely, θ_1, θ_2, and θ_3, but their sum is still a root of the quartic.

If u be greater than θ_1, v and y will each be less than θ_2 and θ_3 by increments aggregating the same amount, and we may at once assume

$$u=\theta_1+\alpha+\beta, \quad v=\theta_2-\alpha, \quad y=\theta_3-\beta.$$

Substituting these values, as in the case of the cubic in the forms in question, and solving for α and β, we may proceed to write all the roots of the form, or the resolvent, in simple radical functions of the coefficients. Evidently the number of possible forms for the equation of the fourth degree is very large, possibly near a hundred. The elimination of u and v in one case, namely,

$$x^4-6(u^2+v^2+y^2)x^2+8(u^3+v^3+y^3)x-3(u^4+v^4+y^4)$$
$$+6(u^2v^2+u^2y^2+v^2y^2)=0,$$

gives a resolvent of the 12th degree. There may be others. As u, v, and y

change for each form the values of α and β have to be determined in each case in order to ascertain the roots of that particular form.

With only one other application of the adjunctive equations, which is introduced here for purposes of analogy hereafter, I take leave of the theory of the biquadratic.

If we consider the form

$$x^4 + A\,x + B = 0\,,$$

the adjunctive equations become, putting a, b, e, j, and the variable y, equal zero,

$$c + f + 1 = 0\,,$$
$$c + d + g + 4 = 0\,,$$
$$2\,d + h + 6 = 0\,.$$

Assuming the two constants d and g equal zero, we have at once

$$c = -4\,, \quad f = 3\,, \quad h = -6\,,$$

and the equation in u and v becomes

$$x^4 - 4\,(u^3 + v^3)\,x + 3\,(u^4 + v^4) - 6\,u\,v^2 = 0\,.$$

Whence, equating,

$$-4\,(u^3 + v^3) = A\,, \quad 3\,(u^4 + v^4) - 6\,u^2v^2 = B\,;$$

or, simplifying,

$$u^2 - v^2 = P,$$
$$u^3 + v^3 = Q\,,$$

which presents an important analogy with the corresponding form

$$u^2 + v^2 = P,$$
$$u^3 + v^3 = Q,$$

already discussed in the case of the cubic equation. It will be observed that these forms involve the general solution of the fourth and third degrees, the first form solving the quartic, the second the cubic. The determination of u and v in radicals has already been given.

Proceeding to the study of the general equation of the fifth degree,

$$x^5 + A\,x^3 + B\,x^2 + C\,x + D = 0\,,$$

the corresponding general equation becomes

$$x^5 + [a\,\Sigma u^2 + b\,\Sigma uv]\,x^3 + [c\,\Sigma u^3 + d\,\Sigma u^2v + e\,\Sigma uvy]\,x^2$$
$$+ [f\,\Sigma u^4 + g\,\Sigma u^3v + h\,\Sigma u^2v^2 + j\,\Sigma u^2vy + k\,uvyz]\,x$$
$$+ l\,\Sigma u^5 + m\,\Sigma u^4v + n\,\Sigma u^3v^2 + p\,\Sigma u^3vy + q\,\Sigma u^2v^2y + r\,\Sigma u^2vyz = 0\,.$$

Assuming

$$x = u + v + y + z,$$

we obtain directly

$$a + c + f + l + 1 = 0\,,$$
$$3a + b + 2c + d + f + g + m + 5 = 0\,,$$
$$4a + 3b + c + 3d + g + h + n + 10 = 0\,,$$
$$6a + 7b + 2c + 4d + e + g + j + p + 20 = 0\,,$$
$$6a + 12b + 8d + 2e + h + 2j + q + 30 = 0\,,$$
$$6a + 27b + 7e + 3j + k + r + 60 = 0\,.$$

There being four independent coefficients, we assume x equal to as many independent variables; but, if the number of coefficients be diminished, we diminish the number of variables correspondingly. If, for example, the coefficient A be assumed equal to zero, then will z be zero; and if B be also zero, we may put y equal zero. If one coefficient and one variable be put equal to zero, obviously only the first five of the above six adjunctive equations will occur; if two coefficients and two variables be each put equal zero, only the first three of the adjunctive equations will appear, and the quintic will be identically satisfied.

We may now deduce a remarkably simple form for the equation of the fifth degree as viewed from several aspects. If A and B be each assumed equal to zero, the general equation becomes the Jerrardian

$$x^5 + Cx + D = 0\,.$$

We may assume y and z equal to zero, the constants $a, b, c, d, e, j, k, p, q, r$ become zero, and the general equation in x, u, and v reduces to

$$x^5 + [f\,\Sigma u^4 + g\,\Sigma u^3v + h\,u^2v^2]\,x + l\,\Sigma u^5 + m\,\Sigma u^4v + n\,\Sigma u^3v^2 = 0\,.$$

The remaining adjunctive equations also become

$$f + l + 1 = 0\,,$$
$$f + g + m + 5 = 0\,,$$
$$g + h + n + 10 = 0\,.$$

Of these six constants f, g, h, l, m, n we may assume m, n, and g equal to zero; whence

$$f = -5, \quad h = -10, \quad l = 4,$$

and the above equation becomes

$$x^5 + [-5(u^4 + v^4) - 10\,u^2v^2]\,x + 4(u^5 + v^5) = 0.$$

Equating with the above Jerrardian,

$$-5(u^4 + v^4) - 10\,u^2v^2 = C, \quad 4(u^5 + v^5) = D;$$

whence, reducing,

$$u^2 + v^2 = P,$$
$$u^5 + v^5 = Q;$$

and the solution of this form for u and v, as in its analogous form for the cubic (6), involves the reduction of the general equation of the fifth degree. This, apparently, is a simplification of the Jerrardian form; for, if we substitute $u + v$ for x in any Jerrardian, it asserts only one relation between the variables, and it is visibly non-decomposable into binomial forms.

Common Resolvent for the Fifth and Sixth Degrees.

In the quintic adjunctive group assume $a, b, d, e, f, g, h, j, k, n, p, q, r$, and y and z equal to zero, whence

$$c = -10, \quad l = 9, \quad m = 15,$$

and the Jerrardian

$$x^5 + Bx^2 + D = 0$$

becomes

$$x^5 - 10(u^3 + v^3)\,x^2 + 9(u^5 + v^5) + 15\,uv(u^3 + v^3) = 0.$$

Whence, by simplifying,

$$\left.\begin{aligned} u^3 + v^3 &= P, \\ u^5 + v^5 + \tfrac{5}{3}\,uv(u^3 + v^3) &= Q. \end{aligned}\right\} \qquad (a)$$

The diminutive group for the general sextic

$$x^6 + Ax + B = 0,$$

for the form

$$x^6 + [a\,\Sigma u^5 + b\,\Sigma u^4v + c\,\Sigma u^3v^2]\,x + d\,\Sigma u^6 + e\,\Sigma u^5v + f\,\Sigma u^4v^2 + g\,u^3v^3 = 0,$$

is

$$a + d + 1 = 0,$$
$$a + b + e + 6 = 0,$$
$$b + c + f + 15 = 0,$$
$$2c + g + 20 = 0.$$

Assuming c, e, f equal to zero, we have

$$a = 9\,, \quad b = -15\,, \quad d = -10\,, \quad g = -20\,;$$

and substituting and equating, as usual

$$9\,(u^5 + v^5) - 15\,uv\,(u^3 + v^3) = A\;; \quad -10\,(u^6 + v^6) - 20\,u^3v^3 = B\,.$$

Whence

$$\left.\begin{aligned} u^3 + v^3 &= P, \\ u^5 + v^5 - \tfrac{5}{3}\,uv\,(u^3 + v^3) &= Q\,. \end{aligned}\right\} \qquad (\beta)$$

These equations (α) and (β) are identical, excepting the change of sign, (α) involving the general solution of the fifth degree, (β) that of the sixth degree. This analogy of the change of sign was pointed out for the third and fourth degrees, where the resolvent for both forms was of the sixth degree, upon which resolvent the solution of both equations depended.

If from (α) we eliminate v, the resolvent is of the twenty-fourth degree, which is directly convertible into the resolvent for (β) by simply changing the sign of the coefficient; from which we conclude, that the solution of the fifth and sixth degrees depends upon a common resolvent equation of the twenty-fourth degree.

NOTE ON THE NINE-POINT CONIC.

By Dr. Maxime Bôcher, Cambridge, Mass.

I find that in the Educational Times for March, 1864, Clifford refers incidentally, in the solution of a problem set by Prof. Sylvester, to the "nine-point conic;" thus showing that this conic, to which I called attention in the March number of the Annals, was then familiar to English mathematicians. I have not been able, however, to find any earlier mention of the subject.

Mr. Holgate, of Clark University, has called my attention to an anonymous note in the Messenger of Mathematics for 1869 (old series), in which the nine-point conic of a *quadrangle* is spoken of, and a similar terminology is suggested for the nine-point circle. I think something may be said in favor of the reverse of this, i. e. in favor of speaking, as I did in my note, of the nine-point conic of a *triangle* which corresponds to a certain *point.* This terminology, to be sure, introduces an unnecessary distinction between the four vertices of the quadrangle; but, on the other hand, it makes many theorems more simple and striking, and brings the nine-point conic into close connection with the theory of the triangle so much elaborated of late. Thus, for example, the maximum ellipse which can be inscribed in a triangle would be the nine-point conic which corresponds to the centre of gravity of the triangle; or, again (a theorem I will not venture to claim as new), *the polar line of the point P with regard to the triangle ABC (considered as a cubic curve) is parallel to the polar of P with regard to the nine-point conic of the triangle ABC which corresponds to P.*

It hardly needs mention, that many of the properties of the nine-point circle can be extended, with but little loss of simplicity, to the nine-point conic by the method of parallel projection. Thus, Feuerbach's theorem will give us:

Any nine-point conic of a given triangle touches the four conics, similar to itself and similarly placed, which are tangent to the three sides of the triangle.

SYMMETRIES OF THE CUBIC AND METHODS OF TREATING THE IRREDUCIBLE CASE.

By Mr. Chas. H. Kummell, Washington, D. C.

The study of the internal structure of quantics is always interesting, and in a higher sense practically useful, especially if we succeed in exhibiting, by a suitable notation, the symmetries which must exist in them. The following study of the cubic has been undertaken with this view, in part, hoping thereby to discover a method of treating the irreducible case, if one root at least is rational. The ordinary definition of the irreducible case is, that the discriminant is negative. The cube roots in Cardan's formula will then be imaginary, although the three roots of the cubic are real, and may be found by the trisection of an angle. I agree, however, with Guido Weichold, who says, in his elaborate article on the irreducible case (American Journal of Mathematics, Vol. I, p. 32), that it is an essential condition for it, that one root be rational, and that some method of approximation, among which he includes the trigonometrical by trisection of an angle, must be used in case of irrational roots. The oldest method of treating the irreducible case is that of Bombelli, which is as follows:—

Let x_1 be one root of the cubic

$$0 = x^3 - px + q\,;$$

then, by Cardan's formula,

$$x_1 = \sqrt[3]{-\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}} + \sqrt[3]{-\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}}\,,$$

which according to Bombelli is

$$= \tfrac{1}{2}x_1 + \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p} + \tfrac{1}{2}x_1 - \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p}\,;$$

and, if $\quad 1^{\frac{1}{3}} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \quad 1^{\frac{2}{3}} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3},$

the other two roots become

$$x_2 = \left(\tfrac{1}{2}x_1 + \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p}\right)1^{\frac{1}{3}} + \left(\tfrac{1}{2}x_1 - \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p}\right)1^{\frac{2}{3}} = -\tfrac{1}{2}x_1 + \tfrac{1}{2}\sqrt{4p - 3x_1^2},$$

$$x_3 = \left(\tfrac{1}{2}x_1 + \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p}\right)1^{\frac{2}{3}} + \left(\tfrac{1}{2}x_1 - \sqrt{\tfrac{1}{4}x_1^2 + \tfrac{1}{3}p}\right)1^{\frac{1}{3}} = -\tfrac{1}{2}x_1 - \tfrac{1}{2}\sqrt{4p - 3x_1^2}.$$

Although this method assumes the previous knowledge of one root, it is as satisfactory and convenient as any method yet proposed. This will be better understood after we have studied the internal relations of the general cubic

$$\begin{aligned} 0 = (x, y)_3 &= ax^3 + bx^2y + cxy^2 + dy^3 \\ &= (xy_1 - x_1y)(xy_2 - x_2y)(xy_3 - x_3y), \end{aligned} \tag{1}$$

in which we suppose the coefficients a, b, c, d to be integers, so that one root at least, for instance $x_1 y_1^{-1}$, may be rational. We have, by Taylor's theorem,

$$(x + \Delta x, y)_3 = (x, y)_3 + \frac{\partial}{\partial x}(x, y)_3 \Delta x + \tfrac{1}{2}\frac{\partial^2}{\partial x^2}(x, y)_3 \Delta x^2 + \tfrac{1}{6}\frac{\partial^3}{\partial x^3}(x, y)_3 \Delta x^3$$

$$= ax^3 + bx^2y + cxy^2 + dy^3$$

$$+ (3\,ax^2 + 2\,bxy + cy^2)\,\Delta x + (3\,ax + by)\,\Delta x^2 + a\,\Delta x^3. \qquad (2)$$

If in this we put $x = -b$, $y = 3a$ and $\Delta x = 3a\,xy^{-1} + b$, it becomes

$$0 = (3a\,xy^{-1}, 3a)_3 = (3ay^{-1})^3 (x, y)_3$$

$$= a(3a\,xy^{-1} + b)^3 + \frac{\partial}{\partial x}(-b, 3a)_3 (3a\,xy^{-1} + b) + (-b, 3a)_3$$

$$= a\,(3a\,xy^{-1} + b)^3 - 3a\,(b^2 - 3ac)\,(3a\,xy^{-1} + b) + a\,(2b^3 - 9\,abc + 27\,a^2d)$$

$$= (3a\,xy^{-1} + b)^3 - 3\,(b^2 - 3ac)\,(3a\,xy^{-1} + b) + 2b^3 - 9\,abc + 27\,a^2d. \qquad (3)$$

If we vary y instead of x, we have

$$0 = (x, y + \Delta y)_3 = (x, y)_3 + \frac{\partial}{\partial y}(x, y)_3 \Delta y + \tfrac{1}{2}\frac{\partial^2}{\partial y^2}(x, y)_3 \Delta y^2 + \tfrac{1}{6}\frac{\partial^3}{\partial y^3}(x, y)_3 \Delta y^3$$

$$= ax^3 + bx^2y + cxy^2 + dy^3$$

$$+ (bx^2 + 2\,cxy + 3\,dy^2)\,\Delta y + (cx + 3\,dy)\,\Delta y^2 + d\,\Delta y^3\,; \qquad (4)$$

which, by placing $x = 3d$, $y = -c$, $\Delta y = c + 3d\,x^{-1}y$, becomes

$$0 = (3d, 3d\,x^{-1}y)_3 = (3dx^{-1})^3 (x, y)_3$$

$$= d\,(c + 3d\,x^{-1}y)^3 + \frac{\partial}{\partial y}(3d, -c)_3 (c + 3d\,x^{-1}y) + (3d, -c)_3$$

$$= (c + 3d\,x^{-1}y)^3 - 3\,(c^2 - 3bd)\,(c + 3d\,x^{-1}y) + 2c^3 - 9\,bcd + 27\,ad^2. \qquad (5)$$

For greater conciseness, let us write

the cubic variant, $\quad 2b^3 - 9abc + 27a^2d = A\,;$ (6)

" quadratic variant, $\quad b^2 - 3ac = A'\,;$ (7)

" " retrovariant, $\quad c^2 - 3bd = D'\,;$ (8)

" cubic " $\quad 2c^3 - 9bcd + 27ad^2 = D.$ (9)

Then we have

$$0 = (3a\,xy^{-1} + b)^3 - 3A'\,(3a\,xy^{-1} + b) + A, \qquad (3')$$

$$0 = (c + 3d\,x^{-1}y)^3 - 3D'\,(c + 3d\,x^{-1}y) + D. \qquad (5')$$

Since by (1)

$$a = y_1 y_2 y_3, \qquad b = -x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3,$$
$$c = x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3, \quad d = -x_1 x_2 x_3; \tag{10}$$

the unknown quantities $3a\,xy^{-1} + b$, $c + 3d\,x^{-1}y$ will have at least one integer value, if x_1 and y_1 are integers.

Assuming now, in (3′)

$$0 = 3a\,xy^{-1} + b + b' + b'', \tag{11}$$

and in (5′)

$$0 = c'' + c' + c + 3d\,x^{-1}y, \tag{12}$$

where b', b'' and c', c'' are real or imaginary conjugates, whose sums are integers, then (3′) and (5′) become

$$0 = (b' + b'')^3 - 3A'(b' + b'') - A, \tag{3''}$$

$$0 = (c' + c'')^3 - 3D'(c' + c'') - D. \tag{5''}$$

We have, then,

$$A = b'^3 + b''^3, \tag{6'}$$

$$A' = b'b'', \tag{7'}$$

$$D' = c'c'', \tag{8'}$$

$$D = c'^3 + c''^3; \tag{9'}$$

and consequently

$$\left.\begin{matrix} b' \\ b'' \end{matrix}\right\} = \sqrt[3]{\tfrac{1}{2}A \pm \sqrt{\tfrac{1}{4}A^2 - A'^3}} = \sqrt[3]{\tfrac{1}{2}A \pm \tfrac{3}{2}a\sqrt{\varDelta_3}}, \tag{13}$$

$$\left.\begin{matrix} c' \\ c'' \end{matrix}\right\} = \sqrt[3]{\tfrac{1}{2}D \pm \sqrt{\tfrac{1}{4}D^2 - D'^3}} = \sqrt[3]{\tfrac{1}{2}D \pm \tfrac{3}{2}d\sqrt{\varDelta_3}}, \tag{14}$$

where $\varDelta_3$ denotes the discriminant. If this is negative, for the determination of either b', b'' or c', c'' a cube root of a complex quantity has to be extracted; and in this consists the whole difficulty of the irreducible case; for, having done this, the roots of the cubic are immediately given by (11) or (12).

Let us now study the mutual relations of the pair b', b'' to the pair c', c''. If (11) gives one root $x_1 y_1^{-1}$, which we suppose rational; then all three roots will be given by the system

$$\begin{aligned} 0 &= 3a\,x_1y_1^{-1} + b + b' + b'', \\ 0 &= 3a\,x_2y_2^{-1} + b + b'\,1^{\frac{1}{3}} + b''\,1^{\frac{2}{3}}, \\ 0 &= 3a\,x_3y_3^{-1} + b + b'\,1^{\frac{2}{3}} + b''\,1^{\frac{1}{3}}. \end{aligned} \tag{15}$$

Similarly from (12) follow the relations

$$
\begin{aligned}
0 &= c'' + c' + c + 3d\,x_1^{-1}y_1,\\
0 &= c''\,1^{-\frac{2}{3}} + c'\,1^{-\frac{1}{3}} + c + 3d\,x_2^{-1}y_2, \qquad (16)\\
0 &= c''\,1^{-\frac{1}{3}} + c'\,1^{-\frac{2}{3}} + c + 3d\,x_3^{-1}y_3.
\end{aligned}
$$

It will be noticed that the cube root of unity factors of b', c' and also of b'', c'' for the same root of the cubic correspond in such a manner that their product is unity. This is in a measure arbitrary, but is adopted because it produces the most perfect symmetry. From (15) we deduce

$$
\begin{aligned}
0 &= a^{-1}b + x_1y_1^{-1} + x_2y_2^{-1} + x_3y_3^{-1},\\
0 &= a^{-1}b' + x_1y_1^{-1} + x_2y_2^{-1}\,1^{-\frac{1}{3}} + x_3y_3^{-1}\,1^{-\frac{2}{3}}, \qquad (17)\\
0 &= a^{-1}b'' + x_1y_1^{-1} + x_2y_2^{-1}\,1^{-\frac{2}{3}} + x_3y_3^{-1}\,1^{-\frac{1}{3}}.
\end{aligned}
$$

Similarly from (16):

$$
\begin{aligned}
0 &= y_3x_3^{-1} + y_2x_2^{-1} + y_1x_1^{-1} + cd^{-1},\\
0 &= y_3x_3^{-1}\,1^{\frac{2}{3}} + y_2x_2^{-1}\,1^{\frac{1}{3}} + y_1x_1^{-1} + c'd^{-1}, \qquad (18)\\
0 &= y_3x_3^{-1}\,1^{\frac{1}{3}} + y_2x_2^{-1}\,1^{\frac{2}{3}} + y_1x_1^{-1} + c''d^{-1}.
\end{aligned}
$$

Then remembering $0 = 1 + 1^{\frac{1}{3}} + 1^{\frac{2}{3}} = 1 + 1^{-\frac{1}{3}} + 1^{-\frac{2}{3}}$, we easily deduce the following:

$$
\begin{aligned}
b'b'' &= b^2 - 3ac = A',\\
b''b &= b'^2 - 3ac', \qquad (19)\\
bb' &= b''^2 - 3ac'',
\end{aligned}
$$

$$
\begin{aligned}
bc + b'c' + b''c'' &= 9\,ad,\\
bc' + b'c'' + b''c &= 0, \qquad (20)\\
bc'' + b'c + b''c' &= 0,
\end{aligned}
$$

$$
\begin{aligned}
c'c &= c''^2 - 3\,b''d,\\
cc'' &= c'^2 - 3\,b'd, \qquad (21)\\
c''c' &= c^2 - 3\,bd = D',
\end{aligned}
$$

Since, therefore,

$$b'c' + b''c'' = 9\,ad - bc,$$

and

$$b'c' \,.\, b''c'' = (b^2 - 3\,ac)(c^2 - 3\,bd) = A'D';$$

the products $b'c' = p'$ and $b''c'' = p''$ will be the roots of the quadratic

$$0 = p^2 - (9\,ad - bc)\,p + A'D'. \tag{22}$$

This remarkable resolvent, which bears such perfect symmetry to the cubic that it is unaffected by exchanging a for d, b for c, x for y, and A' for D', has as far as I am aware never been given. Solving it we have

$$\left.\begin{array}{l} b'c' \\ b''c'' \end{array}\right\} = \left.\begin{array}{l} p' \\ p'' \end{array}\right\} = \tfrac{1}{2}(9\,ad - bc) \pm \tfrac{1}{2}\sqrt{(9\,ad - bc)^2 - 4\,A'D'}$$

$$= \tfrac{1}{2}(9\,ad - bc) \pm \tfrac{1}{2}\sqrt{\varDelta_3}. \tag{23}$$

From (19) and (21) we deduce

$$bA' = bb'b'' = b^3 - 3\,abc = b'^3 - 3\,ab'c' = b''^3 - 3\,ab''c'', \tag{24}$$

$$cD' = cc'c'' = c^3 - 3\,bcd = c'^3 - 3\,b'c'd = c''^3 - 3\,b''c''d. \tag{25}$$

Therefore

$$b' = \sqrt[3]{bA' + 3\,ap'} = \sqrt[3]{b^3 - 3\,a\,(bc - b'c')},$$

$$b'' = \sqrt[3]{bA' + 3\,ap''} = \sqrt[3]{b^3 - 3\,a\,(bc - b''c'')}, \tag{26}$$

and

$$c' = \sqrt[3]{cD' + 3\,p'd} = \sqrt[3]{c^3 - 3\,d\,(bc - b'c')},$$

$$c'' = \sqrt[3]{cD' + 3\,p''d} = \sqrt[3]{c^3 - 3\,d\,(bc - b''c'')}. \tag{27}$$

The coefficients a, b, c, d of the cubic, expressed in terms of the auxiliaries b', b'', c', c'', are as follows:

$$a = \frac{b'^3 - b''^3}{3\,(b'c' - b''c'')} = \frac{1}{3\sqrt{\varDelta_3}}\,(b'^3 - b''^3), \tag{28}$$

$$b = -\frac{b'^2c'' - b''^2c'}{b'c' - b''c''} = -\frac{1}{\sqrt{\varDelta_3}}\,(b'^2c'' - b''^2c'), \tag{29}$$

$$c = \frac{b'c''^2 - b''c'^2}{b'c' - b''c''} = \frac{1}{\sqrt{\varDelta_3}}\,(b'c''^2 - b''c'^2), \tag{30}$$

$$d = -\frac{c''^3 - c'^3}{3\,(b'c' - b''c'')} = -\frac{1}{3\sqrt{\varDelta_3}}\,(c''^3 - c'^3). \tag{31}$$

The cubic takes then the form, which is termed the canonical,

$$0 = (x, y)_3 = \frac{1}{3\sqrt{\varDelta_3}}\,[\,(b'x - c''y)^3 - (b''x - c'y)^3]\,; \tag{32}$$

and the three roots of this are given by

$$0 = b'x_1 - c''y_1 - (b''x_1 - c'y_1); \qquad \therefore\ x_1 = c'' - c', \quad y_1 = b' - b'': \tag{33_1}$$

$$0 = (b'x_2 - c''y_2)\,1^{\frac{1}{3}} - (b''x_2 - c'y_2)\,1^{\frac{2}{3}};\ \therefore\ x_2 = c''\,1^{\frac{1}{3}} - c'\,1^{\frac{2}{3}}, \quad y_2 = b'\,1^{\frac{1}{3}} - b''\,1^{\frac{2}{3}}: \tag{33_2}$$

$$0 = (b'x_3 - c''y_3)\,1^{\frac{2}{3}} - (b''x_3 - c'y_3)\,1^{\frac{1}{3}};\ \therefore\ x_3 = c''\,1^{\frac{2}{3}} - c'\,1^{\frac{1}{3}}, \quad y_3 = b'\,1^{\frac{2}{3}} - b''\,1^{\frac{1}{3}}. \tag{33_3}$$

Substituting $x = c''$ and $y = b'$, and again $x = c'$ and $y = b''$, in (32), we obtain

$$(c'', b')_3 = (c', b'')_3 = \tfrac{1}{3}\,\Delta_3. \tag{34}$$

The quadri-covariant, Hessian or canonizant, is defined

$$(3_2\widehat{)(}x, y)_2 = \begin{vmatrix} 3ax + by\,, & bx + cy \\ bx + cy\,, & cx + 3dy \end{vmatrix}$$

$$= \begin{vmatrix} y^2\,, & xy\,, & x^2 \\ 3a\,, & b\,, & c \\ b\,, & c\,, & 3d \end{vmatrix};$$

or more briefly, $\qquad = (3_2)_2.$* $\qquad$ (35)

Expressing this in terms of the auxiliaries we have

$$(3_2)_2 = -b'b''x^2 + (b'c' + b''c'')\,xy - c'c''y^2$$

$$= -(b'x - c''y)\,(b''x - c'y). \tag{36}$$

The cubicovariant is defined

$$\begin{aligned}(3_3\widehat{)(}x, y)_3 = (3_3)_3 = \quad & (2b^3 - 9\,abc + 27\,a^2d)\,x^3 && = \ Ax^3 \\ & + 3\,(b^2c - 6\,ac^2 + 9\,abd)\,x^2y && + 3\,Bx^2y \\ & - 3\,(bc^2 - 6\,b^2d + 9\,acd)\,xy^2 && - 3\,Cxy^2 \\ & - \ (2c^3 - 9\,bcd + 27\,ad^2)\,y^3 && - Dy^3. \end{aligned} \tag{37}$$

---

* This is an example of a general notation for covariants and invariants of one or more quantics; thus, $(n_p\widehat{)(}x, y)_g$ denotes a covariant of a quantic $(n_1\widehat{)(}x, y)_n$ of the nth order, whose weight in coefficients is p, and which is of the gth degree in form. Also $(n_p, n'_{p'}\widehat{)(}x, y)_g$ denotes a covariant of two quantics, one of the nth order entering with weight p and another of the n'th order entering with weight p', the resulting form being of degree g. If the variables need not be shown, the shorter forms $(n_p)_g$, $(n_1)_n$, $(n_p, n'_{p'})_g$ may be used, and for invariants $g = 0$. The principal quantic may be denoted $(x, y)_n$, as we have done above.

Now, we have from (19), (20), (21), (24), and (25):

$$b'^3 + b''^3 = 2\,bb'b'' + 3\,a\,(b'c' + b''c'') = 2\,b^3 - 9\,abc + 27\,a^2d \;= A,$$
$$b'c''^2 + b''^2c' = b\,(b'c' + b''c'') + 6\,ac'c'' \;= -\,b^2c + 6\,ac^2 - 9\,abd \;= -\,B,$$
$$b'c''^2 + b''c'^2 = c\,(b'c' + b''c'') + 6\,b'b''d \;= -\,bc^2 + 6\,b^2d - 9\,acd = -\,C,$$
$$c''^3 + c'^3 = 2\,cc'c'' + 3\,d\,(b'c' + b''c'') \;= 2\,c^3 - 9\,bcd + 27\,ad^2 \;= D.$$

Therefore

$$(3_3\widehat{)(}x,\,y)_3 = (3_3)_3 = (b'x - c''y)^3 + (b''x - c'y)^3\,; \tag{38}$$

and since

$$[(b'x - c''y)^3 + (b''x - c'y)^3]^2 = [(b'x - c''y)^3 - (b''x - c'y)^3]^2$$
$$+\,4\,(b'x - c''y)^3\,(b''x - c'y)^3,$$

we have Cayley's relation,

$$(3_3)_3^{\,2} = 9\,\Delta_3\,(x,\,y)_3^{\,2} - 4\,(3_2)_2^{\,3}. \tag{39}$$

It is interesting to note the following special values of the cubic and its covariants:

$$(-\,b,\,3\,a)_3 = A,$$
$$(3\,d,\,-\,c)_3 = D,$$
$$(c'',\,b')_3 = (c',\,b'')_3 = \tfrac{1}{3}\,\Delta_3,$$
$$(c'' - c',\,b' - b'')_3 = 0, \tag{40}$$
$$(3_2\widehat{)(}-\,b,\,3\,a)_2 = -\,b'^2\,b''^2 = -\,A'^2,$$
$$(3_2\widehat{)(}3\,d,\,-\,c)_2 = -\,c'^2\,c''^2 = -\,D'^2,$$
$$(3_2\widehat{)(}c'',\,b')_2 = (3_2\widehat{)(}c',\,b'')_2 = 0,$$
$$(3_2\widehat{)(}c'' - c',\,b' - b'')_2 = -\,\Delta_3, \tag{41}$$
$$(3_3\widehat{)(}-\,b,\,3\,a)_3 = -\,b'^6 - b''^6 = -\,A^2 + 2\,A'^3,$$
$$(3_3\widehat{)(}3\,d,\,-\,c)_3 = c'^6 + c''^6 = D^2 - 2\,D'^3,$$
$$(3_3\widehat{)(}c'',\,b')_3 = -\,\Delta_3^{\frac{3}{2}} = -\,(3_3\widehat{)(}c',\,b'')_3,$$
$$(3_3\widehat{)(}c'' - c',\,b' - c'')_3 = -\,2\,\Delta_3^{\frac{3}{2}}. \tag{42}$$

It is also evident that the linear substitutions to reduce the cubic to the canonical form must be

$$x = c''X + c'\,Y, \tag{43}$$
$$y = b'X + b''\,Y; \tag{44}$$

for we have then

$$\begin{aligned}(x, y)_3 &= (c''X + c'Y, b'X + b''Y)_3 \\ &= (c'', b')_3 X^3 + [c'(3\,ac''^2 + 2\,bc''b' + cb'^2) \\ &\qquad\qquad + b''(bc''^2 + 2\,cc''b' + 3\,db'^2)]\,X^2Y \\ &\quad + [c''(3\,ac'^2 + 2\,bc'b'' + cb''^2) \\ &\qquad\qquad + b'(bc'^2 + 2\,cc'b'' + 3\,db''^2)]\,XY^2 + (c', b'')_3\,Y^3 \\ &= \tfrac{1}{3}\,\Delta_3\,(X^3 + Y^3),\end{aligned}$$

by (40), and because each of the middle terms vanishes identically by virtue of (28), (29), (30), and (31). Now, because

$$X = -\frac{b''x - c'y}{b'c' - b''c''} = -\frac{1}{\sqrt{\Delta_3}}\,(b''x - c'y), \tag{45}$$

$$Y = \frac{b'x - c''y}{b'c' - b''c''} = \frac{1}{\sqrt{\Delta_3}}\,(b'x - c''y)\,; \tag{46}$$

we have

$$(x, y)_3 = \frac{1}{3\sqrt{\Delta_3}}\,[(b'x - c''y)^3 - (b''x - c'y)^3]$$

as before.

I shall now discuss some of the principal methods of treating the irreducible case. Of these the most remarkable is, perhaps, that of Guido Weichold, l. c. He employs the same auxiliaries only with different notation, he has

$$b' = -\rho\,, \quad b'' = -\rho',$$
$$c' = \rho_1\,, \quad c'' = \rho_1'.$$

He expresses, in terms of the coefficients of the cubic, certain pairs of quantities, such as

$b'b''$ and $b'c'$, or $c'c''$ and $b'c'$, or $b'b''$ and b'^3, or $c'c''$ and c'^3, etc.,

which have a common factor, and one of which, in the irreducible case, is a complex quantity. If at least one root of the cubic is rational, these complex quantities b', b'', c', c'' must be such that, being added to their respective conjugates, an integer results. It must then be possible to determine

$b' = \overline{b'b''\,|\,b'c'} =$ factor common to $b'b''$ and $b'c'$,

or $\quad c' = \overline{c'c''\,|\,b'c'} =$ " " " $c'c''$ " $b'c'$, etc.,

by the arithmetical process for finding the greatest common divisor. If this process terminates, the last divisor which divides exactly will be a multiple of the common factor sought, for instance, b'; whence we know also its conjugate b'', and the roots of the cubic, result from (15) or (16). Thus we avoid the determination of these auxiliaries by extraction of a cube root of a complex quantity by (13), (14) or (26), (27), which is obviously impossible. There is, however, this difficulty, that the correct quotient cannot be assumed, because the dividend is given in binomial instead of trinomial form. If, then, the process terminates, it is, in reality, a random success. To make this clear, let

$$b' = \tfrac{1}{2}\,a + \tfrac{1}{2}\sqrt{-3\,\beta}\,, \qquad b'' = \tfrac{1}{2}\,a - \tfrac{1}{2}\sqrt{-3\,\beta}\,, \tag{47}$$

$$c' = \tfrac{1}{2}\,\delta \pm \tfrac{1}{2}\sqrt{-3\,\gamma}\,,^{*} \qquad c'' = \tfrac{1}{2}\,\delta \mp \tfrac{1}{2}\sqrt{-3\,\gamma}\,; \tag{48}$$

then $\quad b'b'' = \tfrac{1}{4}\,a^2 + \tfrac{3}{4}\,\beta\,,$

and $\quad b'c' = \tfrac{1}{4}\,a\delta \pm \tfrac{1}{4}\,(a\sqrt{\gamma} \mp \delta\sqrt{\beta})\sqrt{-3} \mp \tfrac{3}{4}\sqrt{\beta\gamma}\,.$

Using $b'b''$ as first divisor we see that the exact quotient of the first division is $a^{-1}\delta$ which will be used if a divides δ exactly; otherwise, the nearest integer will be assumed. But, since the first and third term are merged into one quantity, it is impossible to know either the exact or approximate quotient. This alone does not, however, vitiate the result, provided a correct quotient is used by chance, or otherwise, at some future step. Besides, it is obviously impossible, without some criterion, to know whether the cubic has rational roots or not. In the latter case, Weichold's exceedingly tedious process would never terminate, and this without any indication in the process itself. We need, therefore, a criterion to decide this. The following identity may be used for this purpose:

$$\frac{b'^3 + b''^3}{b' + b''} - b'b'' = (b' + b'')^2 - 4\,b'b'' = (b' - b'')^2.$$

Expressing this by means of (6'), (7'), and (47), we have

$$A a^{-1} - A' = a^2 - 4\,A' = -\,3\,\beta. \tag{49}$$

Similarly, we have for the reciprocal solution of the cubic

$$D\delta^{-1} - D' = \delta^2 - 4\,D' = -\,3\,\gamma. \tag{50}$$

---

* The sign of $\sqrt{-3\,\gamma}$ is indefinite unless we form both pairs of auxiliaries; then their connecting condition, as derived by (20), is

$$9\,ad - bc = \tfrac{1}{2}\,(a\delta - 3\sqrt{\beta\gamma}). \tag{20}$$

This shows α to be an exact divisor of the cubic variant A (also δ of D the cubic retrovariant). If, then, all possible factors of A (or D) are tried by this criterion, and it is not satisfied, we conclude that the cubic cannot have any rational root. It is of course only necessary to try one of these criteria, yet the labor would be very great if each possible factor of A (or D) had actually to be tried, when at most only three can satisfy the criterion. Now it is evident that, if α is a root of the cubic

$$A + 3\,A'\alpha - \alpha^3 = 0\,,$$

then $\alpha \mp n$ will exactly divide the quantity

$$A_{\mp n} = A \pm 3\,A'n \mp n^3\,; \tag{51}$$

so that

$$\frac{A_{\mp n}}{\alpha \mp n} = Q_{\mp n},\ \text{an integer.} \tag{52}$$

Similarly, if

$$D_{\mp n} = D \pm 3\,D'n \mp n^3, \tag{53}$$

then

$$\frac{D_{\mp n}}{\delta \mp n} = R_{\mp n},\ \text{an integer.} \tag{54}$$

Finding, then, also the factors of $A_{-1} = A + 3\,A' - 1$, for example, we can, by comparing them with those of A, exclude a considerable number which it is unnecessary to try by the criterion. If we also find the factors of A_{+1}, and of others of these quantities if necessary, we shall be able to select such factors as satisfy the criterion. There are now three cases:

1) There are no factors of $A_{-n}, \ldots, A_{-1}, A, A_{+1}, \ldots, A_{+n}$ in regular sequence; then we conclude that the cubic can have no rational root.

2) There is but one sequence in these factors; in this case the cubic has one rational root.

3) There are three sequences; which happens if the cubic has three rational roots.

All this is, however, only an application of a well-known method of determining rational roots of any binary quantic, and we see that the knowledge of one root is required, exactly as in Bombelli's method. There is, however, a special advantage in using the criterion in the form given, because it deter-

mines two auxiliaries a, β (or δ, γ) in one operation, whence the roots may be derived by the relations

$$
\begin{aligned}
0 &= 3\,a\,x_1y_1^{-1} + b + a\,, \\
0 &= 3\,a\,x_2y_2^{-1} + b - \tfrac{1}{2}\,a - \tfrac{3}{2}\sqrt{\beta}\,, \\
0 &= 3\,a\,x_3y_3^{-1} + b - \tfrac{1}{2}\,a + \tfrac{3}{2}\sqrt{\beta}\,, \qquad (55) \\
0 &= 3\,d\,y_1x_1^{-1} + c + \delta\,, \\
0 &= 3\,d\,y_2x_2^{-1} + c - \tfrac{1}{2}\,\delta \pm \tfrac{3}{2}\sqrt{\gamma}\,, \\
0 &= 3\,d\,y_3x_3^{-1} + c - \tfrac{1}{2}\,\delta \mp \tfrac{3}{2}\sqrt{\gamma}\,; \qquad (56)
\end{aligned}
$$

or by (33) we have also

$$
\begin{aligned}
x_1y_1^{-1} &= \frac{c'' - c'}{b' - b''} = \mp\sqrt{\frac{\gamma}{\beta}}\,, \\
x_2y_2^{-1} &= \frac{c''\,1^{\frac{1}{3}} - c'\,1^{\frac{2}{3}}}{b'\,1^{\frac{1}{3}} - b''\,1^{\frac{2}{3}}} = \frac{\delta \pm \sqrt{\gamma}}{a - \sqrt{\beta}}\,, \\
x_3y_3^{-1} &= \frac{c''\,1^{\frac{2}{3}} - c'\,1^{\frac{1}{3}}}{b'\,1^{\frac{2}{3}} - b''\,1^{\frac{1}{3}}} = \frac{\delta \mp \sqrt{\gamma}}{a + \sqrt{\beta}}\,; \qquad (57)
\end{aligned}
$$

in which $\sqrt{\beta}$ and $\sqrt{\gamma}$ may be expressed symbolically thus:

$$\sqrt{\beta} = \sqrt{\tfrac{1}{3}\,(A' - Aa^{-1} = 4\,A' - a^2)}\,, \qquad (58)$$

$$\sqrt{\gamma} = \sqrt{\tfrac{1}{3}\,(D' - D\delta^{-1} = 4\,D' - \delta^2)}\,. \qquad (59)$$

It is now evident that any examination of a cubic for the purpose of learning the nature of its roots, cannot avoid the actual determination of one of its rational roots, if it has any; and since after having proved the non-existence of rational roots, Weichold's process cannot terminate and becomes superfluous if there are rational roots, it is in all cases unnecessary to use it; yet it is one of the most remarkable attempts to solve the irreducible case.

The identity

$$\frac{b'^3 - b''^3}{b' - b''} + b'b'' = 4\,b'b'' + (b' - b'')^2 = (b' + b'')^2\,,$$

or

$$\frac{3\,a\sqrt{\Delta_3}}{\sqrt{-3\,\beta}} + A' = 4\,A' - 3\,\beta = a^2\,, \qquad (60)$$

which shows β to be an exact divisor of the discriminant, might be used, but far less conveniently than (49). This method would be closely related to Kendall's given in the American Journal of Mathematics, Vol. I, p. 285.

The method given by Matthiessen in his great work entitled, Grundzuege der antiken und modernen Algebra, p. 390, is no solution of the irreducible case in the sense in which we have considered it above.

I shall now show the application of our formulæ to the solution of the cubic in its different cases.

1) *No rational root:*

$$0 = x^3 - 5\,xy^2 + 3\,y^3.$$

Here we have $A' = 15$, and $A_{-1} = +125$, with factors 5, 25, 125;

$A = +81$, " " 3, 9, 27, 81;

$A_1 = +37$, " " 37;

and since it is impossible to arrange any of these factors into a sequence, there can be no rational root.

2) *One rational root.*

a. Two imaginary roots:

$$0 = x^3 - x^2y - xy^2 - 2\,y^3.$$

Here we have $A' = +4$, $A_{-1} = -54$, with factors 2, 3, 6, 9, 18, 27, 54;

$A = -65$, " " 5, 13, 65,

$A_1 = -76$, " " 2, 4, 19.

The only possible sequence is 6, 5, 4; hence $a = -6 + 1 = -5 = -4 - 1$, and to test this by criterion (49), I use the form

$$\begin{array}{r|r|l} a = -5 & -65 & +13\,; \quad a^2 = 25 \\ & -65 & -4 \qquad -16 \\ \hline & 0 & +9 = +9 = -3\,\beta. \end{array}$$

We have, therefore,

$$b' = -\tfrac{5}{2} + \tfrac{1}{2}\sqrt{9} = -1\,,$$

$$b'' = -\tfrac{5}{2} - \tfrac{1}{2}\sqrt{9} = -4\,;$$

and by (55),

$$0 = 3\,x_1y_1^{-1} - 1 - 1 - 4\,; \qquad \therefore\ x_1y_1^{-1} = 2:$$

$$0 = 3\,x_2y_2^{-1} - 1 + \tfrac{5}{2} - \tfrac{3}{2}\sqrt{-3}\,; \ \therefore\ x_2y_2^{-1} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}:$$

$$0 = 3\,x_3y_3^{-1} - 1 + \tfrac{5}{2} + \tfrac{3}{2}\sqrt{-3}\,; \ \therefore\ x_3y_3^{-1} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

Or, we may solve reciprocally, and have

$$D' = -5\,,\ D_{-1} = +108, \text{ with factors } 2, 3, 4, 9, 18, 27, 54\,;$$

$$D = +124, \quad \text{"} \quad \text{"} \quad 2, 4, 31, 62\,;$$

$$D_1 = +140, \quad \text{"} \quad \text{"} \quad 2, 4, 5, 7, 10, 14, 20, 28, 35, 70.$$

Here the only sequence is 3, 4, 5 ; hence $\delta = 3 + 1 = 4 = 5 - 1$; and testing this by (50), we have

$$\begin{array}{rr|r|rl} \delta = +4 & +124 & +31\,; & \delta^2 = & 16 \\ & +124 & +5 & & +20 \\ \hline & 0 & +36 & = & +36 = -3\,\gamma. \end{array}$$

We have, therefore,

$$c' = 2 - \sqrt{9} = -1\,,$$

$$c'' = 2 + \sqrt{9} = +5\,;$$

and by (56),

$$0 = -6\,y_1x_1^{-1} - 1 + 4\,; \qquad \therefore\ y_1x_1^{-1} = \tfrac{1}{2}\,;$$

$$0 = -6\,y_2x_2^{-1} - 1 - 2 - 3\sqrt{-3}\,; \ \therefore\ y_2x_2^{-1} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}\,;$$

$$0 = -6\,y_3x_3^{-1} - 1 - 2 + 3\sqrt{-3}\,; \ \therefore\ y_3x_3^{-1} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\,;$$

or combining both solutions, we have, by (57),

$$x_1y_1^{-1} = \pm\sqrt{\frac{36}{9}} = \pm 2\,.$$

But the sign criterion (20′) gives

$$\sqrt{\beta\gamma} = \tfrac{2}{3}(1 - 10 + 18) = +6\,;\ \therefore\ \sqrt{\beta} = +\sqrt{-3}, \text{ and } \sqrt{\gamma} = -2\sqrt{-3}\,;$$

hence $$x_1y_1^{-1} = +2\,;$$

also, $$x_2y_2^{-1} = \frac{4 - 2\sqrt{-3}}{-5 - \sqrt{-3}} = \frac{-14 + 14\sqrt{-3}}{28} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\,,$$

$$x_3y_3^{-1} = \frac{4 + 2\sqrt{-3}}{-5 + \sqrt{-3}} = \frac{-14 - 14\sqrt{-3}}{28} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}\,.$$

We may, however, employ resolvent (22) with advantage. We have

$$0 = p^2 + 19\,p - 20\,;$$

hence $$\left.\begin{array}{l} p' \\ p'' \end{array}\right\} = -\tfrac{19}{2} \pm \tfrac{1}{2}\sqrt{361 + 80} = -\tfrac{19}{2} \pm \tfrac{12}{2} = \left\{\begin{array}{r} +\ 1\,, \\ -\ 20\,; \end{array}\right.$$

and by (26) $$b' = \sqrt[3]{-4+3} = -1\,,$$
$$b'' = \sqrt[3]{-4-60} = -4\,;$$

or by (27) $$c' = \sqrt[3]{5-6} = -1\,,$$
$$c'' = \sqrt[3]{5+120} = +5\,.$$

We obtain then the roots from either pair of auxiliaries by (15) or (16). Using, however (33), we have

$$x_1 y_1^{-1} = \frac{+5+1}{-1+4} = +2\,,$$

$$x_2 y_2^{-1} = \frac{-\frac{5}{2} + \frac{5}{2}\sqrt{-3} - \frac{1}{2} - \frac{1}{2}\sqrt{-3}}{+\frac{1}{2} - \frac{1}{2}\sqrt{-3} - \frac{4}{2} - \frac{4}{2}\sqrt{-3}} = \frac{-6+4\sqrt{-3}}{-3-5\sqrt{-3}} = \frac{-42-42\sqrt{-3}}{84}$$
$$= -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}\,,$$

and

$$x_3 y_3^{-1} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\,.$$

b. Two real irrational roots :

$$0 = 12\,x^3 - 4\,x^2y - 14\,xy^2 - 4\,y^3\,.$$

Here we have $A' = 520$;

$A_{-1} = -20169$, with factors 3, 9, 27, 81, 83, 243, etc.

$A = -21728$, " " 2, 4, 7, 8, 14, 16, 28, 32, 56, etc.

$A_1 = -23287$, " " 29, 803, etc.

The only sequence is 27, 28, 29 ; hence $\alpha = 27 + 1 = 28 = 29 - 1$, and by (50), we have

$$\begin{array}{lr|r|rr}
\alpha = +28 & -21728 & -\ 776 & \alpha^2 = & 784 \\
 & -196 & -\ 520 & & -2080 \\
\hline
 & 212 & -1296 & = & -1296 = -3\,\beta. \\
 & 196 & & & \\
\hline
 & 168 & & & \\
 & 168 & & & \\
\hline
 & 0 & & &
\end{array}$$

We have, therefore,

$$b' = 14 + 18\sqrt{-1}\,,$$

$$b'' = 14 - 18\sqrt{-1}\,;$$

and by (55) we have

$$0 = 36\,x_1y_1^{-1} - 4 + 28\,; \qquad \therefore\ x_1y_1^{-1} = -\tfrac{2}{3}\,;$$

$$0 = 36\,x_2y_2^{-1} - 4 - 14 - 18\sqrt{3}\,;\ \therefore\ x_2y_2^{-1} = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,;$$

$$0 = 36\,x_3y_3^{-1} - 4 - 14 + 18\sqrt{3}\,;\ \therefore\ x_3y_3^{-1} = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\,.$$

For the reciprocal solution we have $D' = 148$, $D = 1712$, and find in the same manner $\delta = -4$, $-3\gamma = -576$; and if we wish to satisfy condition (20) we must put

$$c' = -2 + 12\sqrt{-1}\,,$$

$$c'' = -2 - 12\sqrt{-1}\,.$$

To show, also, how Weichold's method is applied, we have

$$9\,ad - bc = -488\,;$$

whence by (23) $\qquad b'c' = -244 + 132\sqrt{-1}\,,$

and, also, $\qquad c'c'' = D' = 148\,.$

We shall have, then, symbolically

$$\overline{-244 + 132\sqrt{-1}\ |\ 148} = kc'\,;$$

whence, using 148 for first divisor,

$$\frac{-244 + 132\sqrt{-1}}{148} = -2 + \sqrt{-1} + \frac{52 - 16\sqrt{-1}}{148}\,;$$

$\therefore\ 52 - 16\sqrt{-1} = 4\,(13 - 4\sqrt{-1}) =$ first remainder.

Dividing this, as simplified, into former divisor, we have

$$\frac{148}{13 - 4\sqrt{-1}} = \frac{52 + 16\sqrt{-1}}{5} = 10 + 3\sqrt{-1} + \frac{2 + \sqrt{-1}}{5}$$

$$= 10 + 3\sqrt{-1} + \frac{6 + \sqrt{-1}}{13 - 4\sqrt{-1}}\,;$$

$\therefore\ 6 + \sqrt{-1} =$ second remainder, and dividing this into former divisor, we have

$$\frac{13 - 4\sqrt{-1}}{6 + \sqrt{-1}} = 2 - \sqrt{-1}\,.$$

Here the process terminates, therefore $6 + \sqrt{-1} = kc'$; and since by (25)

$$\begin{aligned} c'^3 = cD' + 3\, b'c'd &= -2072 + 2928 - 1584\sqrt{-1} \\ &= 856 - 1584\sqrt{-1} \\ &= k^3(198 - 107\sqrt{-1}); \end{aligned}$$

$$\therefore\ k = 2\sqrt{-1} \text{ and } c' = -2 + 12\sqrt{-1}, \text{ as before.}$$

(3). *Three rational roots*

$$0 = x^3 - 3\,x^2y - 60\,xy^2 - 100\,.$$

We have $A' = 189$, $A_{-1} = -3808 = 2.4.7.8.14.16.17.28.32.34\ldots,$

$$\begin{aligned} A &= -4374 = 2.3.6.9.18.27.54.81.162\ldots, \\ A_1 &= -4940 = 2.4.5.10.13.19.20.26.38.52\ldots, \\ A_2 &= -5500 = 2.4.5.10.11.20.22.25.40.44\ldots, \\ A_3 &= -6048 = 2.3.4.6.7.9.12.14.21.24.27\,. \end{aligned}$$

The only sequences that can be formed are (7, 6, 5, 4, 3), (8, 9, 10, 11, 12), (17, 18, 19, 20, 21), (28, 27, 26, 25, 24), therefore we may try $a = -6$, or $= +9$, or $= +18$, or $= -27$. Trying -6 we have

$$\begin{array}{r|r|rr} a = -6 & -4374 & +729 \quad a^2 = & 36 \\ & -4374 & -189 \quad & -756 \\ \hline & 0 & +540 \text{ not} = & -720 \end{array}$$

$\therefore\ a = -6$ does not satisfy.

$$\begin{array}{r|r|rr} a = +9 & -4374 & -486 \quad a^2 = & 81 \\ & -4374 & -189 \quad & -756 \\ \hline & 0 & -675 \quad = & -675 \end{array}$$

$\therefore\ a = +9$, and $0 = 3\,x_1y_1^{-1} - 3 + 9$; therefore, $x_1y_1^{-1} = -2$.

$$\begin{array}{r|r|rr} a = +18 & -4374 & -243 \quad a^2 = & 324 \\ & -4374 & -189 \quad & -756 \\ \hline & 0 & -432 \quad = & -432 \end{array}$$

$\therefore\ a = +18$, and $0 = 3\,x_2y_2^{-1} - 3 + 18$; therefore, $x_2y_2^{-1} = -5$.

$$\begin{array}{r|r|rr} a = -27 & -4374 & +162 \quad a^2 = & 729 \\ & -4374 & -189 \quad & -756 \\ \hline & 0 & -27 \quad = & -27 \end{array}$$

$\therefore\ a = -27$, and $0 = 3\,x_3y_3^{-1} - 3 - 27$; therefore, $x_3y_3^{-1} = +10$.

This example is remarkable for having a persistent false sequence, for we have also 2 a factor of $A_4 = -6578$, and 1 a factor of $A_5 = -7084$. But then 0 cannot be a factor of A_6, which breaks up the sequence. Since, also, 8 is a factor of $A_{-2} = -3248$, 9 a factor of $A_{-3} = -2700$, 10 a factor of $A_{-4} = -2170$, but 11 no factor of $A_{-5} = -1664$, it persists through ten steps.*

The trigonometric method of solving a cubic by the trisection of an angle is usually considered a solution of the irreducible case. It is as proper and as convenient as any known method in the case of three irrational roots, but does not give the true form of the roots in other cases ; yet the relations of the roots to these angles are so remarkable, that some space may be devoted to showing them. We have by analytical trigonometry,

$$0 = 4\cos^3 \tfrac{1}{3}\varphi - 3\cos\tfrac{1}{3}\varphi - \cos\varphi. \tag{61}$$

Comparing this with (31) we have

$$0 = 3\,a\,xy^{-1} + b + 2\sqrt{A'}\cos\tfrac{1}{3}\varphi\,, \tag{62}$$

$$0 = A - 2A'^{\frac{3}{2}}\cos\varphi\,; \tag{63}$$

and therefore

$$b' = \sqrt{A'}\,e^{\frac{1}{3}\phi i},$$

$$b'' = \sqrt{A'}\,e^{-\frac{1}{3}\phi i}. \tag{64}$$

Similarly, by comparing with (5′) we may assume

$$0 = 2\sqrt{D'}\cos\tfrac{1}{3}\psi + c + 3\,d\,yx^{-1}, \tag{65}$$

$$0 = D - 2\,D'^{\frac{3}{2}}\cos\psi\,; \tag{66}$$

and therefore

$$c' = \sqrt{D'}\,e^{\frac{1}{3}\psi i},$$

$$c'' = \sqrt{D'}\,e^{-\frac{1}{3}\psi i}. \tag{67}$$

By (19), (20), (21) there must be the following relations between these angles :

$$\begin{aligned} b\sqrt{A'}\cos\tfrac{1}{3}\varphi &= A'\cos\tfrac{2}{3}\varphi - 3\,a\sqrt{D'}\cos\tfrac{1}{3}\psi\,, \\ -b\sqrt{A'}\sin\tfrac{1}{3}\varphi &= A'\sin\tfrac{2}{3}\varphi - 3\,a\sqrt{D'}\sin\tfrac{1}{3}\psi\,, \end{aligned} \tag{19′}$$

$$\begin{aligned} &bc + 2\sqrt{A'D'}\cos\tfrac{1}{3}(\varphi+\psi) = 9\,ad\,, \\ &b\sqrt{D'}\cos\tfrac{1}{3}\psi + \sqrt{A'D'}\cos\tfrac{1}{3}(\varphi-\psi) + c\sqrt{A'}\cos\tfrac{1}{3}\varphi = 0\,, \\ &b\sqrt{D'}\sin\tfrac{1}{3}\psi + \sqrt{A'D'}\sin\tfrac{1}{3}(\varphi-\psi) - c\sqrt{A'}\sin\tfrac{1}{3}\varphi = 0\,, \end{aligned} \tag{20′}$$

$$\begin{aligned} c\sqrt{D'}\cos\tfrac{1}{3}\psi &= D'\cos\tfrac{2}{3}\psi - 3\,d\sqrt{A'}\cos\tfrac{1}{3}\varphi\,, \\ -c\sqrt{D'}\sin\tfrac{1}{3}\psi &= D'\sin\tfrac{2}{3}\psi - 3\,d\sqrt{A'}\sin\tfrac{1}{3}\varphi\,. \end{aligned} \tag{21′}$$

* We might, however, have saved ourselves the computation and factoring of the A's external to A_{-1}, A, A_1 by using the condition $a_1a_2a_3 = A$, which is only satisfied by 9, 18, 27.

Also, (24) and (25) become

$$bA' = A'^{\frac{3}{2}} e^{\phi i} - 3\,a\,\sqrt{A'D'}\, e^{\frac{1}{3}(\phi + \psi)i} = A'^{\frac{3}{2}}\, e^{-\phi i} - 3\,a\,\sqrt{A'D'}\, e^{-\frac{1}{3}(\phi + \psi)i},$$

$$cD' = D'^{\frac{3}{2}} e^{\psi i} - 3\,d\,\sqrt{A'D'}\, e^{\frac{1}{3}(\phi + \psi)i} = D'^{\frac{3}{2}}\, e^{-\psi i} - 3\,d\,\sqrt{A'D'}\, e^{-\frac{1}{3}(\phi + \psi)i};$$

or
$$bA' = A'^{\frac{3}{2}} \cos \varphi - 3\,a\,\sqrt{A'D'} \cos \tfrac{1}{3}(\varphi + \psi)\,,$$

$$0 = A'^{\frac{3}{2}} \sin \varphi - 3\,a\,\sqrt{A'D'} \sin \tfrac{1}{3}(\varphi + \psi)\,, \tag{24'}$$

$$cD' = D'^{\frac{3}{2}} \cos \psi - 3\,d\,\sqrt{A'D'} \cos \tfrac{1}{3}(\varphi + \psi)\,,$$

$$0 = D'^{\frac{3}{2}} \sin \psi - 3\,d\,\sqrt{A'D'} \sin \tfrac{1}{3}(\varphi + \psi)\,. \tag{25'}$$

These relations suggest many ways of solving the cubic, among which the following four are perhaps the most convenient and elegant :—

(1) Compute

$$\cos \varphi = \frac{A}{2\,A'^{\frac{3}{2}}}\,; \tag{68}$$

then we have

$$x_1 y_1^{-1} = -\tfrac{1}{3}\,a^{-1}\,(b + 2\sqrt{A'} \cos \tfrac{1}{3}\,\varphi)\,,$$

$$x_2 y_2^{-1} = -\tfrac{1}{3}\,a^{-1}\,(b + 2\sqrt{A'} \cos \tfrac{1}{3}\,(\varphi + 2\pi)\,)\,,$$

$$x_3 y_3^{-1} = -\tfrac{1}{3}\,a^{-1}\,(b + 2\sqrt{A'} \cos \tfrac{1}{3}\,(\varphi - 2\pi)\,)\,. \tag{69}$$

(2) Compute

$$\cos \psi = \frac{D}{2\,D'^{\frac{3}{2}}}\,; \tag{70}$$

then we have

$$y_1 x_1^{-1} = -\tfrac{1}{3}\,d^{-1}\,(2\sqrt{D'} \cos \tfrac{1}{3}\,\psi + c)\,,$$

$$y_2 x_2^{-1} = -\tfrac{1}{3}\,d^{-1}\,(2\sqrt{D'} \cos \tfrac{1}{3}\,(\psi + 2\pi) + c)\,,$$

$$y_3 x_3^{-1} = -\tfrac{1}{3}\,d^{-1}\,(2\sqrt{D'} \cos \tfrac{1}{3}\,(\psi - 2\pi) + c)\,. \tag{71}$$

(3) Compute

$$\cos \tfrac{1}{3}(\varphi + \psi) = \frac{9\,ad - bc}{2\sqrt{A'D'}}\,, \tag{72}$$

and
$$\sin \varphi = \frac{3\,a\sqrt{D'}}{A'} \sin \tfrac{1}{3}(\varphi + \psi)\,; \tag{73}$$

then the roots are found by (69) :

or compute
$$\sin \psi = \frac{3\,d\sqrt{A'}}{D'} \sin \tfrac{1}{3}(\varphi + \psi)\,; \tag{74}$$

then the roots are found by (71).

(4) Compute $\frac{1}{3}(\varphi + \psi)$ by (72), φ by (68) or (73), and ψ by (70) or (74). Assume ψ, so that

$$\tfrac{1}{3}\varphi + \tfrac{1}{3}\psi = \tfrac{1}{3}(\varphi + \psi);$$

then we have

$$x_1 y_1^{-1} = \frac{c'' - c'}{b' - b''} = -\frac{\sqrt{D'}\sin\frac{1}{3}\psi}{\sqrt{A'}\sin\frac{1}{3}\varphi},$$

$$x_2 y_2^{-1} = \frac{c''\,1^{\frac{1}{3}} - c'\,1^{\frac{2}{3}}}{b'\,1^{\frac{1}{3}} - b''\,1^{\frac{2}{3}}} = -\frac{\sqrt{D'}\sin\frac{1}{3}(\psi - 2\pi)}{\sqrt{A'}\sin\frac{1}{3}(\varphi + 2\pi)},$$

$$x_3 y_3^{-1} = \frac{c''\,1^{\frac{2}{3}} - c'\,1^{\frac{1}{3}}}{b'\,1^{\frac{2}{3}} - b''\,1^{\frac{1}{3}}} = -\frac{\sqrt{D'}\sin\frac{1}{3}(\psi + 2\pi)}{\sqrt{A'}\sin\frac{1}{3}(\varphi - 2\pi)}. \tag{75}$$

For illustration of this last elegant method I shall solve the cubic already solved above

$$0 = 12\,x^3 - 4\,x^2y - 14\,xy^2 - 4\,y^3.$$

We have here $A' = 520$, $A = -21728$, $9\,ad - bc = -488$, and $D' = 148$, $D = 1712$; and the remaining part of the solution is done by logarithms in the following form :—

$\log\frac{1}{2}A = 4.03599_n$	$\log\frac{1}{2}(9\,ad - bc) = 2.38739_n$	$\log\frac{1}{2}D = 2.93247$
$-\frac{3}{2}\log A' = -4.07400$	$-\frac{1}{2}\log A'D' = -2.44313$	$-\frac{3}{2}\log D' = -3.25539$
$\log\cos\varphi = 9.96199_n$	$\log\cos\frac{1}{3}(\varphi + \psi) = 9.94426_n$	$\log\cos\psi = 9.67708$
$\frac{1}{3}\varphi = 187°\ 52'$	$\frac{1}{3}(\varphi + \psi) = 208°\ 25'$	$\frac{1}{3}\psi = 20°\ 32'$

$\log\sin\frac{1}{3}\varphi = 9.13630_n$	$\log\sin\frac{1}{3}\psi = 9.54500$
$\log\sin\frac{1}{3}(\varphi + 2\pi) = 9.89732_n$	$\log\sin\frac{1}{3}(\psi - 2\pi) = 9.99404_n$
$\log\sin\frac{1}{3}(\varphi - 2\pi) = 9.96676$	$\log\sin\frac{1}{3}(\psi + 2\pi) = 9.80320$
$\frac{1}{2}\log A' = 1.35800$	$\frac{1}{2}\log D' = 1.08513$
$\log\sqrt{A'}\sin\frac{1}{3}\varphi = 0.49430_n$	$\log\sqrt{D'}\sin\frac{1}{3}\psi = 0.63013$
$\log\sqrt{A'}\sin\frac{1}{3}(\varphi + 2\pi) = 1.25532_n$	$\log\sqrt{D'}\sin\frac{1}{3}(\psi - 2\pi) = 1.07917_n$
$\log\sqrt{A'}\sin\frac{1}{3}(\varphi - 2\pi) = 1.32476$	$\log\sqrt{D'}\sin\frac{1}{3}(\psi + 2\pi) = 0.88833$
$\log(-x_1y_1^{-1}) = 0.13583_n$	$x_1y_1^{-1} = +1.3672$
$\log(-x_2y_2^{-1}) = 9.82385$	$x_2y_2^{-1} = -0.6666$
$\log(-x_3y_3^{-1}) = 9.56357$	$x_3y_3^{-1} = -0.3661$

Comparing these with the values above, we notice that they are here obtained in a different order.

ON SALMON'S AND MACCULLAGH'S METHODS OF GENERATING QUADRIC SURFACES.

By Prof. H. B. Newson, Lawrence, Kas.

The modular and umbilical methods of generating surfaces of the second degree invented by MacCullagh and Salmon, respectively, have been before the mathematical world for fifty years, and have been reproduced in every treatise on these surfaces written in this time, yet their substantial identity has not been generally recognized. This substantial identity I proceed to show as follows:

Following the presentation of the subject given in Frost's Solid Geometry, Chap. XV, MacCullagh's method leads to the equation

$$x^2 + y^2 + z^2 = e_3^2\,[\,(x-\alpha)^2 \sec^2\theta_1 + (y-\beta)^2\,]\,, \tag{1}$$

where θ_1 is half the angle between the planes of real circular section; and e_3, the constant ratio, can easily be shown to be equal to the eccentricity of the principal section of the quadric through the medium and least axes (supposing the surface to be an ellipsoid). The planes of real circular section are perpendicular to the principal plane through the greatest and least axes, and $\tan^2\theta_1 = -e_1^2/e_3'^2$. (For this notation see ANNALS, Vol. V, p. 3). Let θ_2 denote half the angle between the planes of imaginary circular section perpendicular to the principal section through the greatest and medium axes. Then $\tan^2\theta_2 = -e_2^2/e_3^2$. Whence it easily follows that $\sec^2\theta_1 = -\tan^2\theta_2$, and $\sec^2\theta_2 = -\tan^2\theta_1$. Now replacing in equation (1) $\sec^2\theta_1$ by $-\tan^2\theta_2$, we have

$$x^2 + y^2 + z^2 = -\frac{e_3^2}{\cos^2\theta_2}\,[\,(x-\alpha)^2 \sin^2\theta_2 - (y-\beta)^2 \cos^2\theta_2\,]\,. \tag{2}$$

Salmon's umbilical method of generating quadrics leads to the equation

$$x^2 + y^2 + z^2 = k\,[\,(x-\alpha)^2 \sin^2\theta_1 - (z-\gamma)^2 \cos^2\theta_1\,], \tag{3}$$

where θ_1 is the same as above; and k, the constant ratio, is easily shown to be equal to $-e_3'^2/\cos^2\theta_1$, where $e_3'^2$ denotes the conjugate eccentricity. Equation (3) then must be written

$$x^2 + y^2 + z^2 = -\frac{e_3'^2}{\cos^2\theta_1}\,[\,(x-\alpha)^2 \sin^2\theta_1 - (z-\gamma)^2 \cos^2\theta_1\,]. \tag{4}$$

Replacing $\tan^2\theta_1$ by its equal $-\sec^2\theta_2$ we get

$$x^2 + y^2 + z^2 = e_3'^2\,[\,(x-\alpha)^2 \sec^2\theta_2 + (z-\gamma)^2\,]. \tag{5}$$

This establishes the identity of the two methods. For e_3^2 according to the ordinary conception of the eccentricity of a conic passes through all real values between zero and infinity. When it becomes negative, it is then the square of what I have elsewhere called the conjugate eccentricity and denoted by e_3'. Hence, in MacCullagh's modular method, if the square of the modulus be conceived to pass through the complete cycle of real values, and our conception enlarged to include imaginary, as well as real, planes of circular section; then Salmon's method appears as a particular case of MacCullagh's; viz.: when e_3^2 is negative. It must be remembered that when e_3^2 changes to $e_3'^2$, θ_1 becomes θ_2, and the planes of real circular section revolve through 90°, so that in (5) y and z should be interchanged.

By comparing equation (2) and (4), it may be argued in the same way that MacCullagh's method is only a particular case of Salmon's. Thus they are mutually inclusive, and there is no reason for regarding one more general than the other.

SOLUTIONS OF EXERCISES.

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SHOW that

$$\sin\theta > \theta - \frac{\theta^3}{3!} + \frac{1}{45}\left[\frac{\theta^5}{2^2} - \frac{\theta^7}{2^9} + \ldots (-)^{m+1}\frac{\theta^{2m+3}}{2^{\frac{1}{2}(m^2+9m+6)}} \pm \ldots\right];$$

the general term being the mth within the brackets. [*W. H. Echols.*]

SOLUTION.

THE general term as stated in the exercise is wrong; it should be

$$(-1)^{n+1}\frac{\theta^{2n-1}}{(2^2-1)(2^4-1)\ldots(2^{2n-2}-1)\,2^{n-1}};$$

where n is the number of the term in the series.

We have Euler's formula

$$\text{Sin}\,\theta = 2^n \sin 2^{-n}\theta \cos\tfrac{1}{2}\theta \cos\tfrac{1}{4}\theta \ldots \cos 2^{-n}\theta.$$

When $n = \infty$ this becomes

$$\text{Sin}\,\theta = \theta\cos\tfrac{1}{2}\theta\cos\tfrac{1}{4}\theta \ldots \textit{ad. inf.}$$

$$= \theta\,(1 - 2\sin^2\tfrac{1}{4}\theta)(1 - 2\sin^2\tfrac{1}{8}\theta)\ldots.$$

Substituting circular measures for sines, we have

$$\text{Sin}\,\theta > \theta\,(1 - 2^{-3}\theta^2)\,(1 - 2^{-5}\theta^2)\,(1 - 2^{-7}\theta^2)\,\ldots\ ad.\ inf.$$

Expanding this binomial product and arranging according to powers of θ, we observe that the coefficient of the nth term is a set of geometrical series whose terms are the reciprocal powers of two, the exponents being respectively the sums of the numbers 3, 5, 7, ... taken $n - 1$ at a time.

Thus, the coefficient of θ^3 is

$$2^{-3} + 2^{-5} + \ldots = \frac{1}{(2^2 - 1)\,2^1}.$$

That of θ^5 is

$$\left.\begin{array}{l} 2^{-8} + 2^{-10} + \ldots \\ + 2^{-12} + 2^{-14} + \ldots \\ \ldots\ldots\ldots\ldots \end{array}\right\} = \frac{1}{(2^2 - 1)\,(2^4 - 1)\,2^2}.$$

That of θ^7,

$$\left.\begin{array}{l} (2^{-15} + 2^{-17} + \ldots) + (2^{-19} + 2^{-21} + \ldots) + \ldots \\ (2^{-21} + 2^{-23} + \ldots) + (2^{-25} + 2^{-27} + \ldots) + \ldots \\ (2^{-27} + 2^{-29} + \ldots) + (2^{-31} + 2^{-33} + \ldots) + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} = \frac{1}{(2^2 - 1)\,(2^4 - 1)\,(2^6 - 1)\,2^3}.$$

Thus, the general form of the series is easily recognized; and we have, finally,

$$\text{Sin}\,\theta > \theta - \frac{\theta^3}{(2^2 - 1)\,2^1} + \frac{\theta^5}{(2^2 - 1)\,(2^4 - 1)\,2^2} - \ldots$$

$$+ (-1)^{n+1}\frac{\theta^{2n-1}}{(2^2 - 1)(2^4 - 1)\ldots(2^{2n-2} - 1)\,2^{n-1}} \pm \ldots$$

[*W. H. Echols.*]

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